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Palindromic Properties of Two-Dimensional Fibonacci Words

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Abstract. Combinatorial properties of $1D$ Fibonacci words is a well studied topic in Formal language theory. In the year 2000, Apostolico et.al. extended the concept of one dimensional Fibonacci words to two dimensional Fibonacci arrays and investigated the number of repetitions of some structures (squares, tandems). In this paper, we investigate the number of distinct Palindromic occurrences in any given Fibonacci array. We also investigate the number of palindromes in the conjugacy class of a Fibonacci array.

Key-words: Fibonacci arrays, Palindromes, Parikh vector, Conjugacy.

1. Introduction

The study of $1D$ Fibonacci words dates back to the 1970's as it has remarkable combinatorial properties. The properties of Fibonacci words both finite as well as infinite have been extensively studied in literature (see [5, 7–11, 21]). It is well known that such properties have their applications in the study of fractal geometry, formal languages, number theory, quasicrystals etc. The notion of $1D$ Fibonacci word was extended to $2D$ Fibonacci array in [4] when considering repetitive substructures in two dimensional arrays.

A two dimensional word/array is a function on $\mathbb{Z} \times \mathbb{Z}$. Recently, the study of $2D$ words has generated great interest in the field of Pattern recognition, string matching, configurations of discrete planes, etc. The notion of two dimensional periodicity of words was introduced by Amir and Benson in [1–3] where the authors provided results analogous to the "periodicity lemma" on one dimensional words. A two dimensional extension of the theorem of Fine and Wilf was given in [14, 22]. Several two-dimensional generalizations of the familiar Lyndon-Schützenberger periodicity theorem for words was discussed in [12]. The authors give a count for the number of finite arrays that are $2D$ primitive. Words with some special properties have been extended to their two dimensional counterparts. Detection of palindromes in an array or an image or a plane, plays an important role due to their symmetrical nature. Some properties of $2D$ palindromes can be found in [6, 17]. The concept of the Lyndon word was extended to two dimensions in [20] and was used to solve $2D$ horizontal suffix-prefix matching among a set of patterns. Two dimensional Sturmian sequences was defined in [24] to study local configurations in a discrete plane and a characterization of such sequences in terms of $2D$ palindromes was given in [6]. The repetitions of squares and tandems for $2D$ Fibonacci arrays was studied in [4].

In this paper, along the lines of [4] we count the number of distinct non-empty palindromes that occur in any given finite Fibonacci word.

The paper is organized as follows: Section 2. recalls some basic notions. Section 3. discusses Fibonacci arrays and some of their basic properties. In Section 4., a list of Fibonacci arrays that are palindromic and also a count of the exact number of distinct non-empty palindromes in any given Fibonacci array are given. Section 5. investigates the conjugacy class of any given Fibonacci array and finds the number of palindromes in it. We end the paper with few concluding remarks.

2. Preliminaries

An alphabet Σ is a finite non-empty set of symbols or letters. Σ^* denotes the set of all words over Σ including the empty word λ. Σ^+ is the set of all non-empty words over Σ. The length of a word $u \in \Sigma^*$ (i.e., the number of symbols in a word) is denoted by $|u|$. A word $u \in \Sigma^*$ is a prefix of the word w if $w = uv$ for some $v \in \Sigma^*$. The reversal of $u = a_1a_2 \cdots a_n$ is defined to be a string $u^R = a_n \cdots a_2a_1$ where $a_i \in \Sigma$ for $1 \leq i \leq n$. A word u is said to be a palindrome or $1D$ palindrome if $u = u^R$. A word w is said to be primitive if $w = u^n$ implies $n = 1$ and $w = u$. For all other concepts in formal language theory and combinatorics on words, the reader is referred to [19]. For $\Sigma = \{a, b\}$, the sequence $\{f_n\}, n \geq 1$ defined recursively by $f_0 = a$, $f_1 = b$, $f_n = f_{n-1}f_{n-2}$ for $n \geq 2$ is called the sequence of Fibonacci words. Moreover, $|f_n| = F(n)$ for $n \geq 1$ where $F(n) = F(n-1) + F(n-1)$ for $n \geq 2$ is the Fibonacci numerical sequence $(F(0) = 1, F(1) = 1)$. Also, $f_n = f_{i_1}f_{i_2} \cdots f_{i_{F(n)}}$ where $i_1, i_2, \cdots, i_{F(n)} \in \{0, 1\}$ and this representation is called the reduced representation of f_n; the binary string $i_1i_2 \cdots i_{F(n)}$ is called the Fibonacci representation of the integer n and is denoted by *FibRep(n)*. The length of this binary string will be $F(n)$.

2.1. Two-dimensional Arrays

We recall certain basic notions and some basic properties pertaining to two dimensional word concepts. For more information, we refer the reader to [6, 12, 13, 15, 17].

Definition 1. *An array or a two-dimensional word,* $w = [w_{i,j}]_{1 \leq i \leq m, 1 \leq j \leq n}$ *of size* (m, n) *over an alphabet* Σ *is defined as the two dimensional rectangular arrangement of letters from the alphabet* Σ*:*

$$\begin{array}{|ccccc|}
\hline
w_{1,1} & w_{1,2} & \cdots & w_{1,n-1} & w_{1,n} \\
w_{2,1} & w_{2,2} & \cdots & w_{2,n-1} & w_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
w_{m-1,1} & w_{m-1,2} & \cdots & w_{m-1,n-1} & w_{m-1,n} \\
w_{m,1} & w_{m,2} & \cdots & w_{m,n-1} & w_{m,n} \\
\hline
\end{array}$$

Given an array p, $|p|_{\text{row}}$ and $|p|_{\text{col}}$ denote the number of rows and number of columns of p, respectively. An empty array, is an array of size $(0,0)$. The set of all arrays over Σ including the empty array λ is denoted by Σ^{**}, whereas Σ^{++} is the set of all non-empty arrays. A subdomain, subpicture or subarray of an array p, denoted by $p[(i,j),(i',j')]$, is the portion of p located at the positions $\{i, i+1, \ldots, i'\} \times \{j, j+1, \ldots, j'\}$, where $1 \leq i \leq i' \leq |p|_{\text{row}}, 1 \leq j \leq j' \leq |p|_{\text{col}}$. In addition, we say that x is a non-empty prefix of p if x is a subarray of p such that $x = p[(1,1),(|x|_{\text{row}},|x|_{\text{col}})]$ where $1 \leq |x|_{\text{row}} \leq |p|_{\text{row}}, 1 \leq |x|_{\text{col}} \leq |p|_{\text{col}}$.

We also recall the following.

Definition 2. *Let w be a $2D$ word.*

1. *If w is of size (m,n), then the reverse image of w denoted by w^R is defined as: $w^R = [w_{m-i+1,n-j+1}]$, $1 \leq i \leq m$, $1 \leq j \leq n$.*
2. *If $w = [w_{ij}]_{1\leq i\leq m, 1\leq j\leq n}$ is of size (m,n), then the transpose of w, denoted by w^T is defined as: $w^T = [u_{ij}]_{1\leq i\leq n, 1\leq j\leq m}$ such that $u_{ij} = w_{ji}$.*
3. *An array w is said to be a $2D$ palindrome if $w = w^R$.*

3. Fibonacci Arrays

In [4], authors have extended the concept of Fibonacci $(1D)$ words to Fibonacci arrays. We recall the following definition.

Definition 3. *Let $\Sigma = \{a,b,c,d\}$. The sequence of $2D$ Fibonacci arrays, $f_{m,n}(m,n \geq 0)$, is defined as follows.*

1. *$f_{0,0} = a$, $f_{0,1} = b$, $f_{1,0} = c$, $f_{1,1} = d$ where some but not all of a,b,c,d might be identical.*
2. *For $m \geq 1$ and $n \geq 2$ (with column-wise reduction),*

$$f_{m,n} = f_{m,n-1} \oplus f_{m,n-2} = \boxed{f_{m,n-1} \quad f_{m,n-2}}$$

and for $m \geq 2$ and $n \geq 1$ (with row-wise reduction),

$$f_{m,n} = f_{m-1,n} \ominus f_{m-2,n} = \boxed{\begin{array}{c} f_{m-1,n} \\ f_{m-2,n} \end{array}}$$

and so on recursively until all the entries are one of $\{f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}\}$.

We illustrate the above definition with the following example.

Example 1. *Let $\Sigma = \{a,b,c,d\}$ and $f_{0,0} = a$, $f_{0,1} = b$, $f_{1,0} = c$, $f_{1,1} = d$. Then*

$$f_{2,3} = \boxed{\begin{array}{c} f_{1,3} \\ f_{0,3} \end{array}} = \boxed{\begin{array}{cc} f_{1,2} & f_{1,1} \\ f_{0,2} & f_{0,1} \end{array}} = \boxed{\begin{array}{ccc} f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,0} & f_{0,1} \end{array}} = \boxed{\begin{array}{ccc} d & c & d \\ b & a & b \end{array}}$$

We can also construct $f_{2,3}$ as

$$f_{2,3} = \boxed{f_{2,2} \quad f_{2,1}} = \boxed{f_{2,1} \quad f_{2,0} \quad f_{2,1}} = \boxed{\begin{array}{ccc} f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,0} & f_{0,1} \end{array}} = \boxed{\begin{array}{ccc} d & c & d \\ b & a & b \end{array}}$$

Another way to find $f_{m,n}$ was given in [18] using *FibRep(m)* and *FibRep(n)*. It was shown that the indices of the entries of $f_{m,n}$ are the ordered pairs of the Cartesian product of the bits of *FibRep(m)* and *FibRep(n)*. Hence the size of the Fibonacci array $f_{m,n}$ is $F(m) \times F(n)$.

Thus, using Fibonacci representation of m and n one can construct any finite $2D$ Fibonacci array quickly as explained below.

$$f_{3,3} = array[f_{\{1,0,1\}\times\{1,0,1\}}] = \boxed{\begin{array}{ccc} f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,0} & f_{0,1} \\ f_{1,1} & f_{1,0} & f_{1,1} \end{array}} = \boxed{\begin{array}{ccc} d & c & d \\ b & a & b \\ d & c & d \end{array}}$$

The $FibRep$ representation of integers m and n, exploits certain properties of rows and columns of $\{f_{m,n}\}$. Throughout this paper we assume $f_{m,n}$, $(m, n = 0, 1, 2, \ldots)$ to be a $2D$ Fibonacci array with $f_{0,0} = a, f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$, over $\Sigma = \{a, b, c, d\}$ with some but not all of a, b, c, d are identical. Before proceeding further, let $\Sigma_1 = \{a, b\}, \Sigma_2 = \{c, d\}$, $\Sigma'_1 = \{a, c\}, \Sigma'_2 = \{b, d\}$ such that $\Sigma = \Sigma_1 \cup \Sigma_2 = \Sigma'_1 \cup \Sigma'_2$.

We have the following observation.

Lemma 1. *For a given $f_{m,n}$, we have*

a. *Any row $($column$)$ of $f_{m,n}$ is a Fibonacci word over either Σ_1 or Σ_2 $(\Sigma'_1$ or Σ'_2 respectively$)$.*

b. *If $\Sigma_1 \neq \Sigma_2$ $(\Sigma'_1 \neq \Sigma'_2)$ then all rows $($columns$)$ of $f_{m,n}$ over Σ_1 (Σ'_1) and all rows $($columns$)$ of $f_{m,n}$ over Σ_2 $(\Sigma'_2$ respectively$)$ are identical.*

c. *If $\Sigma_1 = \Sigma_2$ $(\Sigma'_1 = \Sigma'_2)$, then either all rows $($columns$)$ are identical or a set of rows are identical and are complementary to the set of remaining rows $($columns, respectively $)$ which are identical.*

3.1. Parikh Vector of Fibonacci Arrays

The notion of Parikh vector is a well known concept in formal language theory. This notion was extended to arrays in [23]. For an alphabet $\Sigma = \{a_1, a_2, \cdots, a_k\}$, the Parikh vector of a word $(1D$ or $2D)$ w is defined as $p(w) = [|w|_{a_1}, |w|_{a_2}, \cdots, |w|_{a_k}]$ where $|w|_{a_i}, 1 \leq i \leq k$, denotes the number of occurrences of the letter $a_i, 1 \leq i \leq k$, in the word w. In this section we compute the Parikh vector for any given Fibonacci array $f_{m,n}, m, n \geq 2$. It was shown in [16] that for a $1D$ word w, $|\theta(w)|_b = |w|$ and $|\theta(w)|_a = |w|_b$ where $\theta : a \mapsto b, b \mapsto ba$ is the Fibonacci morphism such that $f_{n+1} = \theta(f_n)$. Thus, one can easily deduce that the Parikh vector of $f_n = [|f_n|_a, |f_n|_b] = [F(n-2), F(n-1)]$.

We now find the Parikh vector of a given Fibonacci array.

Proposition 1. *Let $f_{m,n}$, $m, n \geq 2$ be a $2D$ Fibonacci array over $\Sigma = \{a, b, c, d\}$, such that $a \neq b \neq c \neq d$ with $f_{0,0} = a, f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$. Then, the Parikh vector of $f_{m,n}$ is* $[|f_{m,n}|_a, |f_{m,n}|_b, |f_{m,n}|_c, |f_{m,n}|_d] = [F(m-2)F(n-2), F(m-2)F(n-1), F(m-1)F(n-2), F(m-1)F(n-1)]$.

Proof. By Lemma 1, every row and column of $f_{m,n}$ are 1D Fibonacci words and a row over $\{a, b\}$ cannot have letters from $\{d, c\}$ and vice versa. Since the Fibonacci representation of any integer starts with a '1', every row of $f_{m,n}$ starts with either $f_{1,1} = d$ or $f_{0,1} = b$. Hence, a row, say w, of $f_{m,n}$, starting with the letter b and containing only letters from $\{a, b\}$, is in fact a 1D

Fibonacci word defined by the homomorphism $\theta(a) = b, \theta(b) = ba$ with $f_0 = a$ and will have $|w|_b = F(n-1)$ and $|w|_a = F(n-2)$. Similarly a row, say w', starting with a d and containing only letters from $\{c, d\}$, will have $|w'|_d = F(n-1)$ and $|w'|_c = F(n-2)$. Since the first column of $f_{m,n}$, say v, is a Fibonacci word over $\{d, b\}$ and starts with d, it will have $|v|_d = F(m-1)$ and $|v|_b = F(m-2)$. Hence in $f_{m,n}$ there are $F(m-1)$ rows starting with d and $F(m-2)$ rows starting with b. Thus in total $|f_{m,n}|_d = F(m-1) \times F(n-1)$, $|f_{m,n}|_c = F(m-1) \times F(n-2)$, $|f_{m,n}|_b = F(m-2) \times F(n-1)$ and $|f_{m,n}|_a = F(m-2) \times F(n-2)$. Hence the result. $\square$

4. Palindromic Fibonacci Arrays

The palindromic property of words plays a crucial role in various fields. In this section we find the palindromic Fibonacci arrays and also search and count the number of palindromic sub-arrays in a given finite Fibonacci array $f_{m,n}$.

Definition 4. *A $2D$ palindrome of size $(1, n)$, $n \geq 1$ is called a horizontal palindrome and a $2D$ palindrome of size $(m, 1)$, $m \geq 1$ is called a vertical palindrome.*

We now classify $2D$ palindromes into three different sets.

Definition 5. *A $2D$ Palindrome w is called to be a*

a. *Type I palindrome, if each and every row of w are horizontal palindromes and each and every column of w are vertical palindromes.*

b. *Type II palindrome, if none of the rows of w are horizontal palindromes and none of the columns of w are vertical palindromes.*

c. *Type III palindrome, otherwise.*

We give an example of all the three types of Palindromes below:

$$A = \boxed{\begin{array}{ccc} \mathrm{d} & \mathrm{c} & \mathrm{d} \\ \mathrm{b} & \mathrm{a} & \mathrm{b} \\ \mathrm{d} & \mathrm{c} & \mathrm{d} \end{array}}, B = \boxed{\begin{array}{cccc} \mathrm{d} & \mathrm{c} & \mathrm{d} & \mathrm{c} \\ \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} \\ \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} & \mathrm{c} & \mathrm{d} \end{array}}, C = \boxed{\begin{array}{cccc} \mathrm{d} & \mathrm{c} & \mathrm{d} & \mathrm{c} \\ \mathrm{b} & \mathrm{a} & \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} & \mathrm{c} & \mathrm{d} \end{array}}$$

The palindromes A, B and C are of Type I, Type II and Type III respectively.

Lemma 2. *Palindromes if any, in $\{f_{m,n}\}$ are of*

1. *Type I or Type II only when $f_{0,0} = f_{1,1}$, $f_{0,1} = f_{1,0}$ ((i.e.) $a = d$, $b = c$).*
2. *Type I otherwise.*

Proof. For $1 \leq i \leq m$, let x, y denote the i^{th} and $(m-i+1)^{th}$ rows respectively. Since the rows of $f_{m,n}$ are over either $\{a, b\}$ or over $\{c, d\}$, if $f_{m,n}$ is a palindrome then x and y should be over the same letters, i.e, either over $\{a, b\}$ or $\{c, d\}$ with $x = y^R$. But by Lemma 1, we have either $x = y^c$ when $a = d$ and $b = c$ or $x = y$ otherwise. If $x = y$ then $x = y^R = y = x^R$, which implies that each row is a palindrome. If $x = y^c \neq y$, then $y \neq y^R$ and hence $x \neq x^R$. Thus none of the rows is a palindrome. Similar discussion also holds for columns. Hence if $f_{m,n}$ is a palindrome then it is of Type I or Type II if $(a = d, b = c)$ and Type I otherwise. $\square$

One can observe that the property of Fibonacci arrays $f_{m,n}$ discussed in Lemma 1 also holds true for any subarray of $f_{m,n}$. Thus, we conclude the following.

Corollary 1. *The palindromic subarrays of a given Fibonacci array are also of Type I, except when $f_{0,0} = f_{1,1}, f_{0,1} = f_{1,0}$, in which case some of the palindromic subarrays can be of Type II.*

It was shown in [25] that there are exactly three $1D$ Fibonacci words that are palindromic in nature $(\{a, b, bab\})$. Here we find all $2D$ Fibonacci arrays that are also $2D$ palindromes.

Theorem 1. *Let $\Sigma = \{a, b, c, d\}$ and $f_{0,0} = a$, $f_{0,1} = b$, $f_{1,0} = c$, $f_{1,1} = d$ with $a \neq b \neq c \neq d$. Let P_{2D} be the set of all two dimensional palindromes and let $m, n \geq 0$. Then $\{f_{m,n}\} \cap P_{2D} = \{f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}, f_{1,3}, f_{3,1}, f_{3,3}\}$*

$$= \left\{ a, b, c, d, \boxed{\begin{array}{ccc} d & c & d \end{array}}, \boxed{\begin{array}{c} d \\ b \\ d \end{array}}, \boxed{\begin{array}{ccc} d & c & d \\ b & a & b \\ d & c & d \end{array}} \right\}.$$

Proof. Since $f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}$ are words of size $(1, 1)$, obviously they are palindromes. The arrays $f_{1,3}$ and $f_{3,1}$ are horizontal and vertical palindromes over $\{d, c\}$ and $\{c, a\}$ respectively.

By Lemma 2, palindromes if any in $\{f_{m,n}\}$ are of Type I only. By Lemma 1 every row of $f_{m,n}$ is a $1D$ Fibonacci word and hence none of $\{f_{i,j}, i, j \geq 2,\ i, j \neq 3\}$ can be $2D$ palindromes. Since all rows (columns) of $f_{3,3}$ are horizontal (vertical) palindromes, $f_{3,3}$ is a $2D$ palindrome. $\square$

4.1. Palindromic Subarrays of Fibonacci Arrays

We observed that, when $a \neq b \neq c \neq d$, only seven of the Fibonacci arrays $\{f_{m,n}\}$ are palindromes. In this section, we search for all the distinct palindromic subarrays of a given Fibonacci array. We systematically enumerate them using the structural properties of $2D$ Fibonacci arrays, stated in Lemma 1. We recall the following (see [8, 9, 11]) factorizations of f_n which helps our enumeration process.

Lemma 3. *Let $\{f_n\}$ be the $1D$ Fibonacci sequence with $f_0 = a$ and $f_1 = b$. Then we have the following.*

1. *$f_n = \alpha_n d_n$, $n \geq 3$ where α_n is a palindrome and for $k \geq 1$,*

$$d_n = \begin{cases} ba & : \quad n = 2k \\ ab & : \quad n = 2k+1 \end{cases}$$

2. *$f_n = u_n v_n$, for all $n > 4$, where both u_n and v_n are palindromes that are determined uniquely such that $|u_n| = F(n-1) - 2$ and $|v_n| = F(n-2) + 2$.*

3. *$|\alpha_n|_b = F(n-1) - 1$, $|u_n|_b = F(n-2) - 1$, $|v_n|_b = F(n-3) + 1$*

Note that α_n is the longest palindromic prefix of f_n and u_n is a palindromic prefix of α_n. By their lengths and their positions it is clear that they are distinct. Every time we remove the prefix and the suffix of length one from a palindrome we get another palindrome whose length is reduced by two. By this process we get $F(n)$ palindromic factors of f_n.

Lemma 4. *Let f_n, $n > 2$ be a $1D$ Fibonacci word over $\{a, b\}$, such that $f_n = bx$, $x \in \Sigma^+$. Then the number of palindromic subwords of f_n of the form byb , $y \in \Sigma^+$ is $F(n-1)$.*

Proof. We recall from Lemma 3 that for a given f_n, $n > 2$, $|f_n|_b = F(n-1)$, $|\alpha_n|_b = F(n-1)-1$, $|u_n|_b = F(n-2)-1$, $|v_n|_b = F(n-3)+1$. Note that for any palindrome of the form byb, the letter b is distributed evenly on both sides of the middle symbol of the palindrome. Hence for any palindrome w, if $|w|_b = k$, then there will be $\lceil \frac{k}{2} \rceil$ number of palindromes starting with b. Therefore, we have $\lceil \frac{F(n-1)-1}{2} \rceil + \lceil \frac{F(n-2)-1}{2} \rceil + \lceil \frac{F(n-3)+1}{2} \rceil = F(n-1)$ number of palindromes starting with the symbol b. $\square$

We now investigate the total number of distinct palindromes in any given Fibonacci array.

Theorem 2. *The number of distinct palindromic subarrays of $f_{m,n}$, $m, n > 2$ is $F(m) \times F(n)$.*

We need a sequence of results which helps us in the proof of Theorem 2.

Lemma 5. *$f_{m,n}$ has $2F(m)$ vertical palindromes if a row over $\{a, b\}$ is distinct from a row over $\{c, d\}$ and $2F(n)$ horizontal palindromes if a row over $\{a, c\}$ is distinct from a row over $\{b, d\}$.*

Proof. From Lemma 1, we note that every row of $f_{m,n}$ is either a Fibonacci word over $\{a, b\}$ or $\{c, d\}$ or of the form $e^{F(n)}$ where $e \in \Sigma$. Since Fibonacci words and $e^{F(n)}$ are rich in palindromes, any row of $f_{m,n}$ has $F(n)$ distinct, non-empty palindromic subwords. Also, a row over $\{a, b\}$ is distinct from a row over $\{c, d\}$ and since there are only two distinct rows in $f_{m,n}$ we have $2F(n)$ number of horizontal palindromes. A similar argument over the columns of $f_{m,n}$ proves the second part of the Lemma. $\square$

Lemma 6. *Let $f_{m,n}$ be a Fibonacci array for a given $m, n \geq 2$ that has $2F(m)$ vertical and $2F(n)$ horizontal palindromes. Let $V = \{V_{i,ac}, V_{i,bd}\}, 1 \leq i \leq F(m)$ be the $2F(m)$ vertical palindromes and $H = \{H_{j,ab}, H_{j,cd}\}, 1 \leq j \leq F(n)$ be the $2F(n)$ horizontal palindromes. Let $V_p \in V$ of size $(m', 1)$ and $H_q \in H$ of size $(1, n')$ have a common prefix of size $(1, 1)$ where $m', n' \geq 1$. If any two rows of $f_{m,n}$ over $\{a, b\}$ or $\{c, d\}$ are identical, then it is always possible to find a Type I palindromic subarray of $f_{m,n}$ of size (m', n').*

Proof. Note that V_p and H_q are palindromes. If any two rows of $f_{m,n}$ over $\{a, b\}$ or $\{c, d\}$ are identical, a subword of length l of any such row occuring at a column index j will occur at the same column index in another identical row. Also, since the rows over $\{a, b\}$ and $\{c, d\}$ are generated by the morphisms $\theta(a) = b, \theta(b) = ba$ and $\theta(c) = d, \theta(d) = dc$ respectively, a row over $\{c, d\}$ can be obtained from a row over $\{a, b\}$ by letting $a = c$ and $b = d$. Due to this fact, for every palindromic subword over $\{a, b\}$ of length l' at column index j', there is a palindromic subword over $\{c, d\}$ of the same length, which occurs at the same column index.

Now, let $V_p = v_{p_1} v_{p_2} \cdots v_{p'_m}^T$ where $v_{p_1}, v_{p_2}, \cdots, v_{p'_m} \in \{b, d\}$ or $\{a, c\}$. Now position V_p and H_q at their common top left prefix of size $(1, 1)$. By the above argument we always have palindromic rows in H starting with $v_{p_1}, v_{p_2}, \cdots v_{p'_m}$ of length that is same as that of H_q. By positioning them over $v_{p_1}, v_{p_2}, \cdots v_{p'_m}$, we get a palindromic subarray of $f_{m,n}$ of size (m', n'). $\square$

Proof. (of Theorem 2)

If a row over $\{a, b\}$ is distinct from a row over $\{c, d\}$ then by lemma 5, $f_{m,n}$ has $2F(m)$ vertical palindromes. We have the following cases.

1. If a column over $\{a, c\}$ is distinct from a column over $\{b, d\}$ then by lemma 5, $f_{m,n}$ has $2F(n)$ horizontal palindromes. Then by Lemma 4 we conclude that there are $F(n-2)$,

$F(n-1)$, $F(n-2)$ and $F(n-1)$ number of horizontal palindromes starting with a, b, c, d respectively and $F(m-2)$, $F(m-2)$, $F(m-1)$ and $F(m-1)$ vertical palindromes starting with a, b, c, d respectively. If any two rows of $f_{m,n}$ over $\{a, b\}$ or $\{c, d\}$ are identical, a vertical palindrome having a symbol $s \in \Sigma$ at $(1, 1)$ position, will match only with a horizontal palindrome having the same symbol s at $(1, 1)$ position, we have $F(m-2)F(n-2)$ $(F(m-2)F(n-1), F(m-1)F(n-2), F(m-1)F(n-1))$ possible matchings with the matching prefix as $a(b, c, d$, respectively$)$. Hence by Lemma 6 we have a total of $F(m-2)F(n) + F(m-1)F(n) = F(m)F(n)$ distinct palindromes.

2. If a column over $\{a, c\}$ is identical to a column over $\{b, d\}$ which implies $c = d$ and $a = b$, then the number of distinct vertical palindromes is $F(m)$ and each row of $f_{m,n}$ is either $a^{F(n)}$ or $c^{F(n)}$. Thus by Lemma 4, the total number of vertical palindromes starting with a is $F(m-2)$ and vertical palindromes starting with c is $F(m-1)$. There are $F(n)$ horizontal palindromes starting with a and $F(n)$ horizontal palindromes starting with c and hence by Lemma 6, $F(n)$ palindromes starting with a can be paired with the $F(m-2)$ vertical palindromes starting with a to form palindromic subarrys and similarly $F(n)$ palindromes starting with c can be paired with the $F(m-1)$ vertical palindromes starting with c to form palindromic subarrys. Hence the total number of palindromes is $F(n) \times (F(m-1) + F(m-2)) = F(n) \times F(m)$.

The case when a column over $\{a, c\}$ is identical to a column over $\{b, d\}$ as well as row over $\{a, b\}$ is identical to a row over $\{c, d\}$ is not possible as it implies that $a = b = c = d$. Thus there are exactly $F(m) \times F(n)$ distinct palindromic subarrays in a given Fibonacci array $f_{m,n}$. $\square$

We now give an enumerative example, for a rectangular Fibonacci array and investigate the number of palindromes in it.

Example 2. *Consider* $f_{3,4} = \boxed{\begin{array}{ccccc} d & c & d & d & c \\ b & a & b & b & a \\ d & c & d & d & c \end{array}}$

Horizontal Palindromes → $Vertical Palindromes$ ↓	*d (or) b*	*c (or) a*	*dd (or) bb*	*dcd (or) bab*	*cddc (or) abba*
d (or) c	$\boxed{d}$	$\boxed{c}$	$\boxed{d \; d}$	$\boxed{d \; c \; d}$	$\boxed{c \; d \; d \; c}$
b (or) a	$\boxed{b}$	$\boxed{a}$	$\boxed{b \; b}$	$\boxed{b \; a \; b}$	$\boxed{a \; b \; b \; a}$
$\begin{array}{c} d \\ b \\ d \end{array}$ *(or)* $\begin{array}{c} c \\ a \\ c \end{array}$	$\boxed{\begin{array}{c} d \\ b \\ d \end{array}}$	$\boxed{\begin{array}{c} c \\ a \\ c \end{array}}$	$\boxed{\begin{array}{cc} d & d \\ b & b \\ d & d \end{array}}$	$\boxed{\begin{array}{ccc} d & c & d \\ b & a & b \\ d & c & d \end{array}}$	$\boxed{\begin{array}{cccc} c & d & d & c \\ a & b & b & a \\ c & d & d & c \end{array}}$

We have $F(3) \times F(5) = 3 \times 5 = 15$ *palindromic subarrays.*

5. Conjugates and Palindromes in the Conjugacy Class of $f_{m,n}$

Two words x and y are said to be conjugates if there exists words u,v $\in \Sigma^*$ such that $x = uv, y = vu$. This is an equivalence relation on Σ^* and given a x all the words obtained by cyclic permutations of the letters of x constitute the conjugacy class of x.
We have the following notable theorems on conjugacy class of Fibonacci words. It was shown in [9] that the cyclic permutations of a given $1D$ Fibonacci word f_n are all distinct. It was also shown that such a conjugacy class contains exactly one palindrome if $n+1$ is not a multiple of 3 and zero palindromes otherwise. In this section we define the notion of a conjugacy class of a $2D$ Fibonacci word and investigate the number of palindromes in it.

Definition 6. *Let $u'_1, u'_2, \cdots, u'_m$ and $u_1, u_2, \cdots, u_n$ be respectively the m rows and the n columns of an $m \times n$ 2D array u. The operation, cyclic rotation of k columns, $1 \leq k \leq n$, is denoted by $\circlearrowleft_k^{Col}$ and is defined as the word $u_{n-k+1} \oplus \cdots \oplus u_n \oplus u_1 \oplus u_2 \oplus \cdots \oplus u_{n-k-1} \oplus u_{n-k}$. Similarly the operation, cyclic rotation of k rows, $1 \leq k \leq m$, is denoted by $\circlearrowleft_k^{Row}$ and is defined as the word $u'_{n-k+1} \ominus \cdots \ominus u'_n \ominus u'_1 \ominus u'_2 \ominus \cdots \ominus u'_{n-k-1} \ominus u'_{n-k}$.*

Example 3. *If* $u = \boxed{\begin{matrix} a & c & b \\ b & b & a \end{matrix}}$ *then* $\circlearrowleft_2^{Col}\ u = \boxed{\begin{matrix} c & b & a \\ b & a & b \end{matrix}}$

Definition 7. *Let u be a 2D word of size (m, n). Then the Conjugacy Class of u, denoted as $Conj(u)$ is defined as,*

$$Conj(u) = \left\{ \circlearrowleft_i^{Col} \circlearrowleft_j^{Row}\ u, 1 \leq i \leq n, 1 \leq j \leq m \right\}$$

Example 4. *For u given in Example 3 , $Conj(u)$ is the set,*

$$\left\{ \boxed{\begin{matrix} a & b & b \\ b & a & c \end{matrix}}, \boxed{\begin{matrix} b & a & c \\ a & b & b \end{matrix}}, \boxed{\begin{matrix} b & a & b \\ c & b & a \end{matrix}}, \boxed{\begin{matrix} c & b & a \\ b & a & b \end{matrix}}, \boxed{\begin{matrix} b & b & a \\ a & c & b \end{matrix}}, \boxed{\begin{matrix} a & c & b \\ b & b & a \end{matrix}} \right\}$$

We show that for any given $2D$ Fibonacci array $f_{m,n}$, the number of distinct arrays in its conjugacy class is less than or equal to $F(m) \times F(n)$.

Theorem 3. *The number of distinct of $f_{m,n}$, $m, n > 2$ is ,*

$$Conj(f_{m,n}) = \begin{cases} F(m) & (a = b) \neq (c = d) \\ F(n) & (a = c) \neq (b = d) \\ F(m) \times F(n) & \textit{otherwise} \end{cases}$$

Proof. By Definition 7, for every circular rotation of i columns, $1 \leq i \leq F(n)$ we get a conjugate of $f_{m,n}$ and in turn we can perform a circular rotation of j rows, $1 \leq j \leq F(m)$ on this conjugate, so as to get a total of $F(m) \times F(n)$ conjugates. Since all finite $1D$ Fibonacci words are primitive (see [11]), all conjugates of a $1D$ Fibonacci word are distinct. Thus as long as a row (column) in $f_{m,n}$ is a Fibonacci word over a $\Sigma' \subset \Sigma$ with $|\Sigma'| = 2$ all column (row respectively) rotations of $f_{m,n}$ will be distinct. But when $(a = b) \neq (c = d)$ all columns will be equal and hence the $F(n)$ column rotations do not produce new conjugate elements. Hence we get only $F(m)$ distinct conjugates. Similarly, when $(a = c) \neq (b = d)$ all rows become equal and hence the $F(m)$ row rotations do not produce new conjugate elements. Hence we get only $F(n)$ distinct conjugates. In all other cases we obtain $F(m) \times F(n)$ distinct conjugates. $\square$

We now count the number of palindromes in the conjugacy class of a given $2D$ Fibonacci word $f_{m,n}$. We denote by $\mathcal{P}[Conj(f_{m,n})]$ the number of palindromes in the conjugacy class of $f_{m,n}$.

Theorem 4. *For a given $2D$ Fibonacci word $f_{m,n}$ with $m, n > 2$,*
$$\mathcal{P}[Conj(f_{m,n})] = \begin{cases} 0 & \text{if } m+1 = 3k \text{ or } n+1 = 3l,\ k,l \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let $m, n > 2$. We know that in $f_{m,n}$, any two rows (columns) over the same set of letters are identical except when $(a = d) \neq (b = c)$. Such identical rows (columns) continue to be identical, even after any number of column and/or row rotations. Hence following the argument in Lemma 2, palindromes if any, in the conjugates of $f_{m,n}$, can be of Type I only. This conclusion holds true in all the cases, except when $(a = d) \neq (b = c)$, where, we have seen that, a Type II palindrome occurs for $m = n = 2$ and no other Type II palindrome is possible when $m, n > 2$. Hence we can conclude that, for $m, n > 2$, palindromes if any in the conjugates of $f_{m,n}$ can be of Type I only.

It is well known that for a $1D$ Fibonacci word f_n, there exists a palindrome in its conjugacy class, only when $n + 1$ is not a multiple of 3. Hence, if $n + 1$ is not a multiple of 3, we can guarantee a conjugate (and only one conjugate) of $f_{m,n}$ such that all its rows are palindromes. Assume that this conjugate is obtained by $\circlearrowleft_s^{Col} f_{m,n}$. Similarly, only when $m+1$ is not a multiple of 3, we can guarantee a conjugate (and only one conjugate) of $f_{m,n}$ such that all its columns are palindromes. Let this conjugate be obtained by $\circlearrowleft_t^{Row} f_{m,n}$. Hence when both $m + 1$ and $n + 1$ are not multiples of 3, we get the only conjugate (Type I) of $f_{m,n}$, namely $\circlearrowleft_t^{Row}\circlearrowleft_s^{Col} f_{m,n}$ and hence the claim. □

6. Conclusion

In this paper, we have counted the number of palindromes in $\{f_{m,n}\}, m, n \geq 0$, the 2D Fibonacci sequence over $\{a, b, c, d\}$. We have also counted the number of palindromic subarrays present in a given Fibonacci array $f_{m,n}$. Further, a condition for the conjugacy class of a given array $f_{m,n}$, to have a palindrome is also proved. These results, can support the research in understanding the symmetry in 2D Fibonacci words, which are linked to pattern recognition, image processing and crystallography.

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