

Synchronizing the Shared Resources

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Abstract. We define an operation over the sets of resources assigned to some concurrent processes, and express the synchronization feature of the resources by specific axioms. We define an algebra called "synchronizing resource algebra", and present some of its properties. We define some supermetrics over this algebra. The relationship between supermetrics and additive functions is described, as well as some results related to the complete synchronizing resource algebras and their completions. Finally we discuss about stable resources of a process in this context.

1. Introduction

We refer to the computational resources of synchronizing concurrent processes. The problem is old in computer science. However, there is no way to evaluate, classify and compare the resources of synchronizing concurrent processes. Measuring the quantitative attributes of these resources is essential to understand better the synchronizing concurrent processes. Therefore the main goal of this paper is to introduce metrics for the resource space of the synchronizing concurrent processes. We mention here the basic Brinch Hansen's idea to implement multiprogramming by monitors [3]. Brinch Hansen presents the essential properties needed to implement "structured multiprogramming":

- defining operations over the shared resources;
- mutual exclusion of these operations in time;
- synchronizing primitives which permit the control of process scheduling.

When two processes are concurrent and work in a synchronized way, they use the whole set of resources defining the "critical region" over the common resources. This

aspect can be expressed by defining a special operation Δ over the sets of resources assigned to the involved processes. The operation is similar to the symmetric difference of sets. In set theory, the symmetric difference of two sets is the set of elements which are in one of either set, but not in both; the operation is the set-theoretic equivalent of the exclusive disjunction in Boolean logic. In order to express the synchronization feature of the resource sets, we use the axiom $a\Delta b = (a \vee b)\Delta(a \wedge b)$. For this, we need that $a\Delta b$, $a \vee b$ and $a \wedge b$ are resources for all resources a, b . This equation interprets the resource space as a set algebra, and in this way we can use various notions and results given by Measure Theory and Functional Analysis. Thus we can define $d(a, b) = \mu(a\Delta b)$, where μ is a measure over the resource space. The used operations Δ and \wedge provide the space with a structure of a unitary commutative ring. Considering that we have $\mu(a\Delta b) = \mu(a \vee b) - \mu(a \wedge b)$, the resource space can be organized as a dual algebraic structure of a semigroup (monoid) with respect to operation Δ , and of a lattice with respect to the operations \vee and \wedge . In order to introduce a metric on the resource space, it is enough to consider an increasing additive function f , and then we define $d(a, b) = f(a \vee b) - f(a \wedge b)$. Starting from these notions, we define in a formal way the synchronizing resource space, and introduce the notion of supermetric over such a space. The processes are considered as operators working over the resource space. We study several properties of the space, and define sufficient conditions for stability with respect to resources. We apply the Banach fixpoint result, and present several conditions of metric completeness.

The structure of the paper is as follows. Section 2 presents the synchronizing resource algebra and their properties. Section 3 defines some supermetrics over the synchronizing resource algebra. In Section 4 we present the relationship between the supermetrics and additive functions. Some results related to the completion of the synchronizing resource algebras are presented in Sections 5 and 6. Then we discuss the stable resources of a process. Conclusion and references end the paper.

2. Synchronizing Resource Algebras

Definition 1. A *synchronizing resource algebra* is a particular lattice-ordered semigroup system $(S, \cdot, \vee, \wedge, u)$ where:

1. (S, \cdot, u) is a semigroup with unit u (monoid);
2. (S, \vee, \wedge, u) is a lattice with the least element u - if S has a zero z , $z \neq u$ and z is the greatest element;
3. for every $a, b, c \in S$ we have:

$$(i) \quad ab = (a \vee b)(a \wedge b),$$

$$(ii) \quad a(b \vee c) = ab \vee ac,$$

$$(iii) \quad a(b \wedge c) = ab \wedge ac.$$

The usual order in a synchronizing resource algebra $(S, \cdot, \vee, \wedge, u)$ is the order induced by its lattice: $x \leq y$ iff $x = x \wedge y$.

Let S be a synchronizing resource algebra and $R \subseteq S$. R is a synchronizing resource subalgebra of S if R is a synchronizing resource algebra with respect to the operations of S restricted to R , and R (as a lattice) is a sublattice of S .

We provide some examples of synchronizing resource algebras:

1. $S = \{f \mid f : [0, 1] \rightarrow [0, 1]\}$, where $(f \wedge g)(x) = \min(f(x), g(x))$, $(f \vee g)(x) = \max(f(x), g(x))$ and $(fg)(x) = f(x)g(x)$ for every $x \in [0, 1]$. u and z are the constant functions 1 and 0.
2. All the distributive lattices with z and $u < z$ where $ab = a \vee b$.
3. $(\mathbb{N}^*, \cdot, \vee, \wedge, 1)$, where \cdot is the usual multiplication over the natural numbers, $x \vee y = \text{lcm}(x, y)$ and $x \wedge y = \text{gcd}(x, y)$. In this case $x \leq y$ whenever y is a multiple of x . Hence, the order induced by the subjacent lattice of this synchronizing resource algebra is the same with the "natural" preorder induced by the semigroup operation (i.e. $x \sqsubseteq y$ iff there is $t \in S$ such that $y = xt$).
4. The real numbers $x \geq 1$ together with a special symbol ∞ , considering the usual multiplication and order over the real numbers, and $z = \infty$.
5. $([0, 1], \cdot, \preceq)$, considering the usual multiplication over the real numbers, and $x \preceq y \Leftrightarrow y \leq x$.
6. Let \mathcal{C} be a family of subsets of X closed under finite union and intersection, and containing X . Then $(\mathcal{C}, \cup, \cap, \emptyset)$ is a synchronizing resource algebra where $z = X$.
7. Let $(S, \cdot, \vee, \wedge, u)$ be a synchronizing resource algebra, and $X \neq \emptyset$. We define $S^X = \{f \mid f : X \rightarrow S\}$ together with the following operations over S^X : $\forall f, g \in S^X$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $(f \vee g)(x) = f(x) \vee g(x)$, $(f \wedge g)(x) = f(x) \wedge g(x)$, and $u : X \rightarrow S$, $u(x) = u, \forall x \in X$. We also define the relation $f \leq g \Leftrightarrow f(x) \leq g(x), \forall x \in X$, and $z : X \rightarrow S$, $z(x) = z, \forall x \in X$, where z is zero in S . Then $(S^X, \cdot, \leq, \vee, \wedge, u)$ is a synchronizing resource algebra.

If $f \in S^X$, the support of f is $\text{supp}(f) = \{x \in X \mid f(x) \neq u\}$. Let $E(X, S) = \{f \in S^X \mid \text{supp}(f) \text{ is finite}\}$. Since for all $f, g \in E(X, S)$ we have $f \cdot g, f \vee g, f \wedge g \in E(X, S)$, then $E(X, S)$ is a synchronizing resource subalgebra. As a particular case, if $X = \mathbb{N}$, then $S^{\mathbb{N}} = \{(x_n) \mid (x_n) \text{ is a sequence of elements of } S\}$. The structure $(S^{\mathbb{N}}, \cdot, \vee, \wedge, u)$ is a synchronizing resource algebra. Moreover, for $X = \{1, 2, \dots, n\}$ we get $S^X = S^n$. It results that $(S^n, \cdot, \leq, \vee, \wedge, u)$ is a synchronizing resource algebra.

3. Supermetrics over Synchronizing Algebras

Let $(S, \cdot, \vee, \wedge, u)$ be a synchronizing resource algebra. We would like to have over S something similar to a module function over \mathbb{R} . Thus we define a metric over S having similar properties.

Definition 2. A function $d : S \times S \rightarrow [0, \infty)$ is a *supermetric over S* iff it satisfies the following conditions:

1. $d(xz, yz) = d(x \vee y, u) - d(x \wedge y, u), \forall x, y, z \in S,$ (PSM)
2. if $d(x, y) = 0$, then $x = y$.

If $d : S \times S \rightarrow [0, \infty)$ satisfies only the first condition, then d is called a pseudo-supermetric. This is the reason why we denote the first condition by PSM.

We present some properties of a pseudo-supermetric. Many proofs are not included just in order to emphasize the results that could be relevant to the synchronizing processes and resources. Some proofs and other technical details could be found in [5].

Proposition 1. If $d : S \times S \rightarrow [0, \infty)$ is a pseudo-supermetric, then we have:

1. $d(xz, yz) = d(x, y)$, for all $x, y, z \in S$;
2. $d(xy, y) = d(x, u)$, for all $x, y \in S$;
3. $d(x, y) = d(y, x)$, for all $x, y \in S$;
4. $d(xy, u) = d(x, u) + d(y, u)$, for all $x, y \in S$;
5. $d(x \vee y, x \wedge y) = d(x, y)$, for all $x, y \in S$;
6. $d(x, y) = d(x \vee y, y) + d(x \wedge y, y)$, for all $x, y \in S$;
7. $x < y$ implies $d(x, u) \leq d(y, u)$;
moreover, if d is a supermetric, then $x < y$ implies $d(x, u) < d(y, u)$;
8. $E(S) \subseteq \{x \in S \mid d(x, u) = 0\}$; if d is a supermetric, then $E(S) = \{u\}$.

Proposition 2. If $d : S \times S \rightarrow [0, \infty)$ is a pseudo-supermetric on S , then we have $d(x, u) + d(y, u) = d(x \vee y, u) + d(x \wedge y, u)$, and $d(x, x) = 0$ for all $x, y \in S$.

Corollary 1. If $d : S \times S \rightarrow [0, \infty)$ is a pseudo-supermetric on S , then we have $d(x, y) = d(x, u) + d(y, u) - 2d(x \wedge y, u) = 2d(x \vee y, u) - d(x, u) - d(y, u)$, for all $x, y \in S$.

It is easy to see that each pseudo-supermetric induces a valuation over synchronizing resource algebra S . Recall that if (L, \vee, \wedge) is a lattice, then $v : (L, \vee, \wedge) \rightarrow (R, +)$ is a valuation if $v(a \vee b) + v(a \wedge b) = v(a) + v(b)$, for all $a, b \in L$. Moreover, if v is strictly increasing, then (L, \vee, \wedge) is modular [2]. Therefore, if we have a supermetric over a synchronizing resource algebra, then the lattice is modular. Moreover, we

can prove that a synchronizing resource algebra with a supermetric is a distributive lattice.

Proposition 3. If we have a supermetric d on $(S, \cdot, \vee, \wedge, u)$, then the subsequent semigroup (S, \cdot) is cancellative (i.e., if $xa = ya$ then $x = y$ for any $a \in S$).

Corollary 2.

- If we have a finite synchronizing resource algebra with a supermetric, then its subsequent semigroup is a commutative group;
- if we have a supermetric d on $(S, \cdot, \vee, \wedge, u)$, then the subsequent lattice (S, \vee, \wedge) is cancellative (i.e., if $x \vee a = y \vee a$ and $x \wedge a = y \wedge a$, then $x = y$).

By this corollary and Bergman's theorem [2], we get the following result:

Theorem 1. If we have a supermetric d on S , then the subsequent lattice (S, \vee, \wedge) is distributive.

Proposition 4. If $d : S \times S \rightarrow [0, \infty)$ is a pseudo-supermetric, then, for all $x, y, z \in S$, we have

1. $d(xz \vee yz, u) \leq d(x \vee z, u) + d(z \vee y, u)$,
2. $d(xz \wedge yz, u) \geq d(x \wedge z, u) + d(z \wedge y, u)$.

Proposition 5. Every pseudo-supermetric is a pseudo-metric, and every supermetric is a metric.

Proof. Let d be a pseudo-supermetric on S . Then PSM implies $d(x, x) = 0$ for all $x \in S$. Also, $d(x, y) = d(y, x)$ for all $x, y \in S$. Let $x, y, z \in S$. By Proposition 4(1) we have: $d(xz \vee yz, u) \leq d(x \vee z, u) + d(z \vee y, u)$. By Proposition 1(4) and axiom 3(ii), we get $d(x \vee y, u) + d(z, u) \leq d(x \vee z, u) + d(z \vee y, u)$. Multiplying by 2, we get an equivalent inequality $2d(x \vee y, u) - d(x, u) - d(y, u) \leq [2d(x \vee z, u) - d(x, u) - d(z, u)] + [2d(y \vee z, u) - d(y, u) - d(z, u)]$. According to Corollary 1, we get $d(x, y) \leq d(x, z) + d(z, y)$. Therefore, d is a pseudo-metric on S . Moreover, if d is a supermetric, then $d(x, y) = 0$ implies $x = y$, and so d is a metric. \square

Proposition 6. If $d : S \times S \rightarrow [0, \infty)$ is a pseudo-supermetric, then for all $x, y, z \in S$ we have

$$\max\{d(x \vee z, y \vee z), d(x \wedge z, y \wedge z)\} \leq d(x, y)$$

Proposition 6 says that the lattice operations are contractions for any pseudo-supermetric over synchronizing resource algebra. This implies that these operations are uniformly continuous [8]. On the other hand, every pseudo-supermetric is invariant with respect to the semigroup operation, and so this operation is also uniformly continuous. Consequently, any synchronizing resource algebra with a pseudo-supermetric is a uniform synchronizing resource algebra. If d is a supermetric over a synchronizing resource algebra S , the continuity of the lattice operations ensures the compatibility among the lattice order and the convergence generated by d .

Proposition 7. If d is a supermetric on S , and $\{a_n\}, \{b_n\} \subset S$ are two sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$, and $a_n \xrightarrow{d} a$ and $b_n \xrightarrow{d} b$, then $a \leq b$.

Corollary 3. If d is a supermetric on S , then every interval $[a, b]$ is a d -closed set, for all $a, b \in S$ with $a \leq b$.

Proposition 8. If $(x_n) \subset S$ is d -Cauchy, then $\{d(x_n \vee x, u)\}$ and $\{d(x_n \wedge x, u)\}$ are convergent, for all $x \in S$.

4. Additive Functions over Resources

In this section we present a one-to-one correspondence between supermetrics on synchronizing resource algebras and strict increasing additive functions. Let $(S, \cdot, \vee, \wedge, u)$ be a synchronizing resource algebra. A function $f : S \rightarrow \mathbb{R}$ is called additive if $f(xy) = f(x) + f(y)$ for all $x, y \in S$. It is easy to see that $f(u) = 0$ for every additive function f . Therefore, if $f : S \rightarrow \mathbb{R}$ is additive and increasing, then f is positive. Let \mathcal{PSM} be the set of all pseudo-supermetrics on S , \mathcal{SM} be the set of all supermetrics on S , \mathcal{F} be the set of all increasing additive functions on S , and \mathcal{F}_s the set of all strict increasing additive functions on S .

Since λf_1 and $f_1 + f_2$ are in \mathcal{F} for all $\lambda > 0$ and $f_1, f_2 \in \mathcal{F}$, then \mathcal{F} is a cone [2]. In a similar way, we get that $\mathcal{F}_s, \mathcal{PSM}$ and \mathcal{SM} are cones. Let $f : S \rightarrow [0, \infty)$ and $d_f : S \times S \rightarrow [0, \infty)$ defined by $d_f(x, y) = f(x \vee y) - f(x \wedge y)$.

Proposition 9. If $f \in \mathcal{F}$, then $d_f \in \mathcal{PSM}$ and $f(x) = d_f(x, u)$ for all $x \in S$. Moreover, if $x, y \in S$ are comparable, then $d_f(x, y) = |f(x) - f(y)|$.

Proof. Since f is increasing, we have $d_f(x, y) \geq 0$. Let $x, y, z \in S$. Then $d_f(xz, yz) = f(xz \vee yz) - f(xz \wedge yz) = f((x \vee y)z) - f((x \wedge y)z) = f(x \vee y) + f(z) - f(x \wedge y) - f(z) = d_f(x, y)$. On the other hand, we have $d_f(x, u) = f(x \vee u) - f(x \wedge u) = f(x) - f(u) = f(x)$. Therefore $d_f(xz, yz) = d_f(x, y) = d_f(x \vee y, u) - d_f(x \wedge y, u)$, $\forall x, y, z \in S$, i.e., d_f satisfies PSM, and so it is a pseudo-supermetric. \square

Theorem 2. $\mathcal{F} \simeq \mathcal{PSM}$, i.e. there is a linear bijective function $\Phi : \mathcal{F} \rightarrow \mathcal{PSM}$.

Proof. Let $\Phi : \mathcal{F} \rightarrow \mathcal{PSM}$ defined by $\Phi(f) = d_f$, for all $f \in \mathcal{F}$. According to Proposition 9, we have that Φ is well-defined. If $\Phi(f_1) = \Phi(f_2)$, then $f_1(x) = d_{f_1}(x, u) = d_{f_2}(x, u) = f_2(x)$, for all $x \in S$. Thus, Φ is one-to-one. Let $d \in \mathcal{PSM}$ and let $f : S \rightarrow [0, \infty)$ defined by $f(x) = d(x, u)$, for all $x \in S$. By Proposition 1, we have that $f \in \mathcal{F}$. Also $d_f(x, y) = f(x \vee y) - f(x \wedge y) = d(x \vee y, u) - d(x \wedge y, u) = d(x, y)$, for all $x, y \in S$. Hence $\Phi(f) = d$, and Φ is onto.

Let $\lambda > 0$ and $f_1, f_2 \in \mathcal{F}$. Since $d_{f_1+f_2} = d_{f_1} + d_{f_2}$ and $d_{\lambda f} = \lambda d_f$, then Φ is linear (i.e. it is additive and positive homogeneously). \square

Theorem 3. $\mathcal{F}_s \simeq \mathcal{SM}$.

As a consequence, we can reformulate Proposition 3 and Theorem 1.

Theorem 4. If $\mathcal{F}_s \neq \emptyset$, then the semigroup (S, \cdot) is cancellative, and the lattice (S, \vee, \wedge) is distributive.

Let $(\mathbb{N}^*, \cdot, \vee, \wedge, 1)$, where \cdot is the usual multiplication, $x \vee y = \text{lcm}(x, y)$ and $x \wedge y = \text{gcd}(x, y)$. Since the function $f : \mathbb{N}^* \rightarrow [0, \infty)$, $f(n) = \ln(n)$ is additive and strict increasing, it follows that $d_f(n, m) = \ln \frac{\text{lcm}(n, m)}{\text{gcd}(n, m)}$ is a supermetric on \mathbb{N}^* , and the lattice $(\mathbb{N}^*, \vee, \wedge)$ is distributive.

Proposition 10. If $f \in \mathcal{F}$, then f is d_f -uniformly continuous.

Proposition 11. If $\mathcal{F}_s \neq \emptyset$, then $x^2 \wedge y^2 \leq xy \leq x^2 \vee y^2$, for all $x, y \in S$.

Proof. Let $x, y \in S$. Since $\mathcal{F}_s \neq \emptyset$, the lattice $(\mathbb{N}^*, \vee, \wedge)$ is distributive, and then $(x^2 \vee y^2) \wedge xy = (x^2 \wedge xy) \vee (y^2 \wedge xy) = x(x \wedge y) \vee y(y \wedge x) = (x \vee y)(x \wedge y) = xy$. Then $xy \leq x^2 \vee y^2$. Analogous, we prove $x^2 \wedge y^2 \leq xy$. \square

Proposition 12. If $\mathcal{F}_s \neq \emptyset$, then $(x \wedge y)^n = x^n \wedge y^n$ and $(x \vee y)^n = x^n \vee y^n$, for all $x, y \in S$, and $n \in \mathbb{N}^*$.

5. Complete Synchronizing Resource Algebras

Let $(S, \cdot, \vee, \wedge, u)$ be a synchronizing resource algebra and $f \in \mathcal{F}_s$.

We consider the following assertions:

Conditional Complete condition:

(CC) Every non-empty upper bounded subset of S has a supremum;

σ -Complete condition:

(σ C) Every ascending sequence of S has a supremum, and every descending sequence of S has an infimum;

Metric Complete condition:

(MC) Every d_f -Cauchy sequence is d_f -convergent;

Ascending Sequences Convergence condition:

(ASC) If $\{x_n\} \subseteq S$ is an upper bounded ascending sequence, then there is $x = \bigvee_n x_n$ and $x_n \xrightarrow{d_f} x$.

Cantor condition:

(CA) If $\{[a_n, b_n]\}$ is a descending sequence of intervals of S such that $d_f(a_n, b_n) \rightarrow 0$, then there exists $c \in S$ such that $\bigcap_n [a_n, b_n] = \{c\}$.

Bounded to Compact condition:

(B2C) If $A \subseteq S$ is d_f -closed and upper bounded, then A is d_f -compact.

Sequential Scott Continuity condition:

(SSC) If $\{x_n\} \subseteq S$ is an ascending sequence such that there is $x = \bigvee_n x_n$, then $f(x) = \bigvee_n f(x_n)$.

Scott Continuity condition:

(SC) If D is a directed subset of S such that there is $x = \vee D$, then $f(x) = \vee f(D)$.

Hard Scott Continuity condition:

(HSC) For every subset A of S such that there is $x = \vee A$, then $f(x) = \vee f(A)$.

Directed Bounded condition:

(DB) For every directed subset D of S , there is an ascending sequence of elements of D which has the same upper bounds as D .

By [2] (Chap.V, Th.15), we have that (MC) implies (CC) & (ASC). By the same theorem, we also have that (σ C) & (ASC) implies (MC). However, if $\mathcal{F} \neq \emptyset$, then (σ C) is false. Indeed, if $f \in \mathcal{F}$ and (σ C) is true, then, for any $x \in S$, $x \neq u$, the sequence $\{x^n\}$ is ascending. Then, according to (σ C), there is $y \in S$ such that $x^n \leq y$ for all $n \in \mathbb{N}$. Therefore, $nf(x) = f(x^n) \leq f(y)$ for all $n \in \mathbb{N}$; contradiction. It is easy to see that (HSC) implies (SC) and (SC) implies (SSC). The following theorems express finally sufficient conditions for (MC).

Theorem 5. *Let $f \in \mathcal{F}_s$. Then we have:*

1. (CC) & (SSC) implies (ASC);
2. (ASC) implies (SSC);
3. (SSC) & (DB) implies (SC).

In the proof of “(SSC) & (DB) implies (SC)” we do not use the hypothesis that D is a directed set. If we consider another condition, then we get a new result.

Hard Bounded condition:

(HB) For every subset D of S , there is an ascending sequence of elements of D which has the same upper bounds as D .

Theorem 6. (SSC) & (HB) implies (HSC).

Theorem 7. *Let $f \in \mathcal{F}_s$ and $A \subset S$ such that there is $x = \vee A$. Then $f(x) = \vee f(A)$ if only if $x \in cl_{d_f} A$.*

Proof. It is easy to see that $f(y) \leq f(x)$ for all $y \in A$. Then $f(x) = \vee f(A)$ iff for every $\varepsilon > 0$, there is $y \in A$ such that $f(x) < f(y) + \varepsilon$. But $f(x) < f(y) + \varepsilon$ iff $d_f(x, y) < \varepsilon$. Therefore $f(x) = \vee f(A)$ iff for every $\varepsilon > 0$, $A \cap B_{d_f}(x, \varepsilon) \neq \emptyset$ \square

Theorem 8. (ASC) implies (CA).

Proof. Let $\{[a_n, b_n]\}$ be a descending sequence of intervals of S such that $d_f(a_n, b_n) \rightarrow 0$. Since $\{a_n\}$ is an ascending sequence and $a_n \leq b_0$, for all $n \in \mathbb{N}$, by (ASC) there is $c = \vee_n a_n$ and $a_n \xrightarrow{d_f} c$. Because $a_n \leq a_{n+p} \leq b_{n+p} \leq b_n$, for all $p \in \mathbb{N}$,

by Proposition 7 we obtain $a_n \leq c \leq b_n$, for all $n \in \mathbb{N}$. Therefore $c \in \bigcap_n [a_n, b_n]$. If $c' \in \bigcap_n [a_n, b_n]$, then $c \wedge c', c \vee c' \in \bigcap_n [a_n, b_n]$ and $0 \leq f(c \vee c') - f(c \wedge c') \leq f(b_n) - f(a_n) = d_f(a_n, b_n)$, for all $n \in \mathbb{N}$. Hence $f(c \vee c') = f(c \wedge c')$. Then $c \vee c' = c \wedge c'$ and, then $c = c'$. Therefore, there exists $c \in S$ such that $\bigcap_n [a_n, b_n] = \{c\}$. \square

Theorem 9. *(CC) & (HSC) implies (B2C).*

Theorem 10. *If (B2C) and $f^{-1}([0, \alpha])$ is upper bounded for all $\alpha > 0$, then (MC).*

Proof. Let $\{x_n\}$ a d_f -Cauchy sequence. Then there is $\alpha > 0$ such that $f(x_n) \leq \alpha$, for all n . By hypothesis the set $\{x_n\}$ is upper bounded. By (B2C), there is (x_{n_k}) a subsequence of (x_n) , and there is $x \in S$ such that $x_{n_k} \xrightarrow{d_f} x$. Then $x_n \xrightarrow{d_f} x$. \square

Corollary 4. *If (CC), (HSC) and $f^{-1}([0, \alpha])$ is upper bounded for all $\alpha > 0$, then (MC).*

6. Completion by an Additive Algebra

If the metric space (S, d_f) is complete (namely condition (MC)), we have several metric and algebraic properties similar to those of \mathbb{R} .

An important property is (B2C); a similar property characterizes the finite-dimensional normed spaces. In our framework, the boundness is given by the lattice order. On the other hand, we can get more results if we work with a synchronizing resource algebra satisfying (MC). Starting from an arbitrary synchronizing resource algebra S together with an increasing additive function f over S , and using a standard procedure of completion for metric spaces, we can build a synchronizing resource algebra S_1 and a strictly increasing additive function f_1 over S_1 such that the metric space (S_1, d_{f_1}) satisfy (MC), and S is d_{f_1} -dense in S_1 .

If $(S, \cdot, \vee, \wedge, u)$ is a synchronizing resource algebra, and $f \in \mathcal{F}_s$, then the pair (S, f) is called an additive synchronizing resource algebra.

Definition 3. Let (S, f) be an additive synchronizing resource algebra. An additive algebra (S_1, f_1) is called *the completion of (S, f)* iff

1. (S_1, f_1) satisfies (MC),
2. there is an isomorphism $\phi : S \rightarrow \phi(S) \subseteq S_1$ of synchronizing resource algebras,
3. $\phi(S)$ is d_{f_1} -dense in S_1 .

Theorem 11. *Let (S, f) be an additive synchronizing resource algebra. Then there is a completion (S_1, f_1) of (S, f) . Moreover, any two completions of (S, f) are isomorphic.*

6.1. Stable Resources

We discuss here about processes with stable resources. Let $(S, \cdot, \vee, \wedge, u)$ be a synchronizing resource algebra, and $g : S \rightarrow S$ a process over the resources S . A resource $x \in S$ is a stable resource of a process g if $g(x) = x$.

We suppose that the conditions (CC) and (HB) are satisfied, and let \mathcal{F}'_s the set the functions of \mathcal{F}_s which satisfy the conditions (SSC) and (α) . Then, for each $f \in \mathcal{F}'_s$, the space (S, d_f) satisfies (MC). In fact, we can renounce to these four conditions if we work with a complete synchronizing resource algebra (e.g., the completion of (S, f)). Using the Banach fixpoint principle we get the following result.

Theorem 12. *If there are $n, m \in \mathbb{N}^*$ such that $m < n$ and $[xy]^m [g(x)g(y)]^n \leq [x \vee y]^{2m} [g(x) \wedge g(y)]^{2n}$ for all $x, y \in S$, then the resource x is a unique stable resource of process g . Moreover, for any resource $x_0 \in S$ and $f \in \mathcal{F}'_s$, the sequence $x_n = g(x_{n-1})$ satisfies $x_n \xrightarrow{d_f} x$.*

7. Conclusion

The aim of this paper is to define an abstract model describing how some resources can be shared among concurrent processes. We introduce the notion of a synchronizing resource algebra. Such an algebra combines a semigroup and a lattice with some coherence conditions. Subsequently we define a supermetric on the set of resources underlying such an algebra. The properties of a distance function called supermetrics could be expressed in terms of the semigroup and lattice operations. Several results of these supermetrics are proved. The relevance of this model are coming from certain properties which could establish the principles of shared resources.

The approach is also interesting from a mathematical perspective; we define the notion of a pseudo-supermetric on the synchronizing resource algebra, and characterize the supermetrics as the strict, increasing additive functions from such a structure to reals. We also investigate various notions of completions of our structures using the supermetrics, and show that each structure endowed with a strict, increasing additive function (which corresponds to a unique supermetric) has a completion, and any two such completions are isomorphic.

We consider this paper as a first step of modelling the resources shared by several processes in a complex system. Further work is related to the synchronizing processes which appear in the concurrency theory and process algebra.

Acknowledgement. This work has been partially supported by the research grant CEEEX 133/2006.

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